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AN UPPER BOUND ON THE SIZE OF THE SNAKE-IN-THE-BOX

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A snake in a graph is a simple cycle without chords. We give an upper bound on the size of a snake $\mathbf{S}$ in the n-dimensional cube of the form $|\mathbf{S}| \leq 2^{n-1}(1-n^{1/2}/89+O(1/n))$.

1. Introduction

The n-cube $\mathbf{Q}^n$ can be considered as the graph whose vertices are indexed by the binary n-tuples and such that two vertices are adjacent if and only if the corresponding n-tuples differ in exactly one position. A *snake* in a graph is a simple cycle without chords. Let $\mathbf{S}$ denote a snake of the longest size in $\mathbf{Q}^n$. The problem of determining the size $|\mathbf{S}|$ of $\mathbf{S}$ was first studied by Kautz [8], and was initially motivated by the design of error-checking codes for analog to digital conversion. It has since been shown to be connected to various applications, see [2] and references therein. The study of $|\mathbf{S}|$ has been investigated by a number of authors: the order of magnitude of $|\mathbf{S}|$ was first established by Evdomikov [6], who proved the existence of λ such that $|\mathbf{S}| \geq \lambda 2^n$. The current best lower bound on λ is due to Abbot and Katchalski [2] who obtain $\lambda \geq 0.30$. Regarding upper bounds, it is straightforward to obtain $|\mathbf{S}| \leq 2^{n-1} n/(n-1)$, and this was improved by several people [1, 3, 7, 10] to $|\mathbf{S}| \leq 2^{n-1}$. Since then, a number of successive improvements were obtained, [9, 5, 4, 12], the latest being due to Snevely [11] who proved:

$$|\mathbf{S}| \leq 2^{n-1}\left(1-\frac{1}{20n-41}\right) \qquad \text{for } n \geq 12.$$

We shall improve on this for growing n by showing:

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Theorem 1.1. *For all n,*

$$|\mathbf{S}| \leq 1+2^{n-1}\frac{6n}{6n+\frac{1}{6\sqrt{6}}n^{1/2}-7} \leq 2^{n-1}\left(1-\frac{1}{89n^{1/2}}+O\left(\frac{1}{n}\right)\right).$$

2. Preliminaries and sketch of proof

Let the vertex set of $\mathbf{Q}^n$ be indexed by $\{0,1\}^n$. We shall consider each edge to be labeled by an element of $[n]=\{1,2,\cdots,n\}$; e.g. we shall write $A \overset{e}{—} B$ to mean that adjacent vertices A and B correspond to n-tuples that differ in position e.

It will also be convenient to us to identify *labels*, i.e. elements of $[n]$, with the n-tuples of weight 1 and apply addition in $\{0,1\}^n$ (componentwise addition modulo 2) to both vertices and edges. For example we shall allow ourselves to write $A \overset{e}{—} B$ alternatively as $A+e=B$ or $A+B=e$. So that no confusion occurs, vertices will always be denoted by uppercase letters and edges labeled by lowercase letters.

Segments are subpaths of $\mathbf{S}$, e.g. $A \overset{e}{—} B \overset{f}{—} C$, and are denoted by greek lowercase letters. If σ is a segment, we shall allow ourselves the freedom of writing $V\in\sigma$ and $e\in\sigma$ to mean that V is a vertex of σ and e labels an edge of σ respectively. Context should leave no ambiguity. If A and B are vertices of $\mathbf{S}$, denote by $d_{\mathbf{S}}(A,B)$ the distance between A and B in $\mathbf{S}$, i.e. the length of the shortest segment of $\mathbf{S}$ joining A and B.

It may be convenient to keep in mind the following characteristic property of $\mathbf{S}$: if σ is a segment of $\mathbf{S}$ of size $2 \leq |\sigma| \leq |\mathbf{S}|-2$, then there exist two distinct elements of $[n]$ that each label an odd number of edges of σ.

Let $\mathbf{T}$ be the subgraph of $\mathbf{Q}^n$ induced by the set of vertices not in $\mathbf{S}$. By the *degree* $\delta(V)$ of a vertex $V\in\mathbf{T}$, we shall mean the degree of V in $\mathbf{T}$, i.e. the number of vertices of $\mathbf{T}$ adjacent to V.

Our basic strategy is to study the total degree in $\mathbf{T}$, i.e. the quantity

$$\Delta = \sum_{V\in\mathbf{T}} \delta(V).$$

We have:

Lemma 2.1. $n2^{n-1}=|\mathbf{S}|(n-1)+\Delta/2$.

Proof. $n2^{n-1}$ is the total number of edges in $\mathbf{Q}^n$. There are $|\mathbf{S}|$ edges in $\mathbf{S}$. There are $|\mathbf{S}|(n-2)$ edges with one vertex in $\mathbf{S}$ and one vertex in $\mathbf{T}$. There are $\Delta/2$ edges in $\mathbf{T}$. ∎

Any lower bound on Δ will therefore yield an upper bound on $|\mathbf{S}|$.

Sketch of the main argument and plan of the paper.

We are looking for a lower bound on Δ. The first thing we shall do is show that $\mathbf{T}$ must contain vertices of high degree. More precisely, we shall show that any segment of $\mathbf{S}$ of length 7 contains a vertex V such that some neighbour $N \in \mathbf{T}$ of V has degree $\delta(N) \geq n/4$. This first argument is the object of Section 3. Unfortunately, this in itself is not enough to yield a good evaluation of Δ because, as we go over all segments of length 7 of $\mathbf{S}$, we may produce the same vertices of $\mathbf{T}$ of high degree (linear in n) many times.

So the next thing we do is introduce a second argument. The latter says essentially that whenever the first argument of Section 3 produces the same vertex N of high degree several times, this in turn must produce more vertices of $\mathbf{T}$ of high degree.

This second argument hinges on a crucial feature of the vertices of high degree uncovered in Section 3 which can be formulated as follows:

it is possible to construct a set $\mathcal{S}$ of vertex-disjoint segments σ of $\mathbf{S}$ (which we call heavy segments) such that, for each $\sigma \in \mathcal{S}$, there exists $v \in [n]$ which labels both an edge of σ and an edge $O — N$ where $O \in \sigma$ and $N \in \mathbf{T}$ with $\delta(N) \geq n/4$. Furthermore, the number of segments in $\mathcal{S}$ is linear in $|\mathbf{S}|$. This is formalized in Section 4.

The way the segments of $\mathcal{S}$ can produce more vertices of $\mathbf{T}$ of high degree is the object of Section 5.

Section 6 then studies the resulting overall situation which can be summarized as follows. Either the first argument produces many vertices of $\mathbf{T}$ of high degree, and hence a good lower bound on Δ: or not, but then the second argument takes over. The lower the estimation of Δ given by one argument, the higher the estimation given by the other. A minimal unconditional lower bound on Δ is obtained when both arguments yield the same estimation of Δ which is shown to be of order of magnitude $|\mathbf{S}|\sqrt{n}$. This yields theorem 1.1.

3. Looking for vertices of T with high degree

We shall make repeated use of the following easy fact.

Lemma 3.1. *Let A, B, C, D, be distinct vertices of $\mathbf{T}$ such that A is adjacent to B, C and D. Then*

$$\delta(A) + \delta(B) + \delta(C) + \delta(D) \geq n$$

In particular, one of the vertices A, B, C, D has degree greater than or equal to $n/4$.

Proof. For every label $e \in [n]$, $A+e, B+e, C+e, D+e$ may not simultaneously lie on the snake, because they are distinct and $A+e$ is adjacent to the three others. ∎

The following lemma is the main result of this section.

Lemma 3.2. *Suppose* $|\mathbf{S}| \geq 9$. *Let* $V_0 \overset{e_1}{—} V_1 \overset{e_2}{—} V_2 \cdots \overset{e_7}{—} V_7$ *be a segment of length 7 of the snake* $\mathbf{S}$. *There exists a vertex* $N \in \mathbf{T}$, $i \in \{0,1,\cdots,7\}$, *and* $j \in \{1,2,\cdots,7\}$ *such that* $(i,j) \neq (0,7)$, $(i,j) \neq (7,1)$ *and*

- $\delta(N) \geq n/4$.
- N *is a neighbour of* V_i *and* $N + V_i = e_j$.

The proof of Lemma 3.2 consists of finding, among vertices of the form $V_i + e_j$, 4 vertices satisfying Lemma 3.1. We do this through some intermediate lemmas. It will be implicit in the remainder of the section that $|\mathbf{S}| \geq 9$.

Lemma 3.3. *Let* $X \overset{x}{—} V_0 \overset{e_1}{—} V_1 \overset{e_2}{—} V_2 \overset{e_3}{—} V_3 \overset{e_4}{—} V_4 \overset{y}{—} Y$ *be a segment of* $\mathbf{S}$. *Suppose* $e_1 \neq e_4$. *Then one of the following holds.*

(i) *There is a vertex of the form* $V_i + e_j$, $0 \leq i \leq 4$, $1 \leq j \leq 4$, *which is in* $\mathbf{T}$ *and of degree* $\geq n/4$.

(ii) $x = e_4$ *and* $y = e_1$.

(iii) $x = e_3$ *and* $y = e_2$.

Proof. $(V_1+e_4)+(V_0+e_4) = e_1, (V_1+e_4)+(V_4+e_2) = e_3, (V_1+e_4)+(V_4+e_3) = e_2$, so that V_1+e_4 is adjacent to $V_0+e_4, V_4+e_2, V_4+e_3$. Similarly, V_3+e_1 is adjacent to $V_4+e_1, V_0+e_2, V_0+e_3$. It is readily checked that if neither $x = e_4$ and $y = e_1$ nor $x = e_3$ and $y = e_2$, then either $V_0+e_4, V_1+e_4, V_4+e_2, V_4+e_3$ are all in $\mathbf{T}$, or $V_0+e_2, V_0+e_3, V_3+e_1, V_4+e_1$ are all in $\mathbf{T}$; so that Lemma 3.1 applies to one of those two sets of four vertices. (See figure 1). ∎

Lemma 3.4. *Let* $X \overset{x}{—} V_0 \overset{e_1}{—} V_1 \overset{e_2}{—} V_2 \overset{e_3}{—} V_3 \overset{e_4}{—} V_4 \overset{y}{—} Y \overset{z}{—} Z$ *be a segment of* $\mathbf{S}$. *Suppose* $e_1 \neq e_4$. *Suppose* $x = e_4$ *and* $y = e_1$. *Then the segment*

$$V_0 \overset{e_1}{—} V_1 \overset{e_2}{—} V_2 \overset{e_3}{—} V_3 \overset{e_4}{—} V_4 \overset{y}{—} Y \overset{z}{—} Z$$

satisfies conclusion (i) *of Lemma 3.3.*

Proof. We need only check that neither $y = e_1$ and $z = e_2$ (ii) nor $e_1 = e_4$ and $z = e_3$ (iii) hold.

We have supposed $e_1 \neq e_4$, so that (iii) does not hold. Furthermore, we have $X + Z = x + e_1 + e_2 + e_3 + e_4 + y + z = e_2 + e_3 + z$. Hence, because X and Z are nonadjacent vertices of $\mathbf{S}$, $z \neq e_2$. Therefore (ii) does not hold. ∎

Lemma 3.5. *Let* $X \overset{x}{—} V_0 \overset{e_1}{—} V_1 \overset{e_2}{—} V_2 \overset{e_3}{—} V_3 \overset{e_4}{—} V_4 \overset{y}{—} Y \overset{z}{—} Z$ *be a segment of* $\mathbf{S}$. *Suppose* $e_1 \neq e_4$. *Suppose* $x = e_3$ *and* $y = e_2$. *Then one of the vertices* $V_0 + z$, $V_1 + z$, $V_2 + z$, $Z + e_4$, $Z + e_3$ *is in* $\mathbf{T}$ *and of degree* $\geq n/4$.

Proof. $V_1 + z$ is clearly adjacent to $V_0 + z$ and $V_2 + z$. Besides, $(V_1+z)+(Z+e_3) = e_4$ and $(V_1+z)+(Z+e_4) = e_3$, so that $V_1 + Z$ is also adjacent to $Z + e_3$ and $Z + e_4$. We

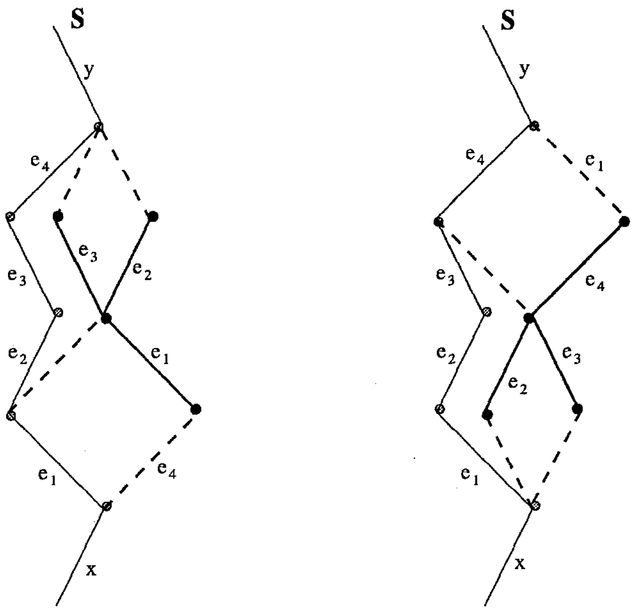

Fig. 1. Two potential vertices of $\mathbf{T}$ of degree 3

have $V_4+Z=e_2+z$, $V_1+Z=e_3+e_4+z$ and $X+Z=e_1+e_4+z$: from which we deduce, since Z is neither equal nor ajacent to X, V_1 and V_4, that $z \neq e_i$ for $i=1,2,3,4$. Hence, either $V_0+z, V_1+z, V_2+z, Z+e_4$ are all in $\mathbf{T}$ or $V_0+z, V_1+z, V_2+z, Z+e_3$ are all in $\mathbf{T}$, and Lemma 3.1 applies to one of those two sets of four vertices. ∎

Proof of Lemma 3.2.

- If $e_1 \neq e_4$, then either Lemma 3.3 or Lemma 3.4 or Lemma 3.4 yields a vertex V_i+e_j of the proper form, for $i \in \{0, \cdots 6\}$ and $j \in \{1, \cdots, 6\}$.
- If $e_1 = e_4$, then $V_0+V_5=e_2+e_3+e_5$ implies $e_5 \neq e_2$. Therefore we can apply the above argument to the segment V_1 — ⋯ — V_7 instead of V_0 — ⋯ — V_6. We obtain, again through Lemmas 3.3, 3.4, 3.5, a vertex V_i+e_j of the proper form, for $i \in \{1, \cdots 7\}$ and $j \in \{2, \cdots, 7\}$. ∎

4. Heavy segments of S

We now reformulate the main result of the preceding section, i.e. Lemma 3.2, in a form more suitable for further study. For this we introduce the following definition.

Definition. A *heavy segment* or *h-segment* of $\mathbf{S}$ is a triple (σ, N, v) where

- σ is a segment of length 10 of $\mathbf{S}$

$$V_{-5} \overset{e_{-5}}{\text{---}} V_{-4} \cdots V_{-1} \overset{e_{-1}}{\text{---}} O \overset{e_1}{\text{---}} V_1 \cdots V_4 \overset{e_5}{\text{---}} V_5$$

- N is a vertex of $\mathbf{T}$ such that $\delta(N) \geq n/4$
- $v \in [n]$ is a label such that N is adjacent to O and $O+N=v$: furthermore, $V_i+v \in \mathbf{S}$ for some $i \in \{-5,\cdots,-1,1,\cdots,5\}$.

Remark. This last condition means that v labels either an edge of σ, or one of the two edges of $\mathbf{S}$ not in σ but adjacent to one of the endpoints of σ.

We shall say that two h-segments (σ,N,v) and (σ',N',v') are *disjoint* if the segments σ and σ' have no common vertex. (We might have $N=N'$, though).

From now on we shall lighten notation by refering to an h-segment simply by σ, and shall refer to the associated vertex and label by $N(\sigma)$ and $v(\sigma)$.

A consequence of Lemma 3.2 is the following.

Lemma 4.1. *There exists a set $\mathscr{S}$ of pairwise disjoint h-segments of* $\mathbf{S}$ *of cardinality*

$$|\mathscr{S}| \geq \frac{|\mathbf{S}|-10}{18}.$$

Proof. If $|\mathscr{S}| \geq 11$, apply Lemma 3.2 to any segment of length 7. We obtain a vertex O_1 which has a neighbour $N \in \mathbf{T}$ such that $\delta(N) \geq n/4$ and such that $N+O_1$ labels an edge of $\mathbf{S}$ at most six edges away from O_1. O_1 is therefore the center of an h-segment σ_1 of $\mathbf{S}$. Then, if there are at least 19 edges in $\mathbf{S}$ that are not in σ_1, choose another segment of length 7, whose nearest endpoint is eleven edges away from O_1. Lemma 3.2 yields again the center O_2 of an h-segment σ_2, disjoint from σ_1 and such that $d_{\mathbf{S}}(O_1,O_2) \leq 18$. Repeat the procedure and construct a sequence $\sigma_1,\cdots,\sigma_i$ of disjoint h-segments. As long as there remains in $\mathbf{S}$ a segment of 19 edges that does not contain any edge of any of the σ_i's, we can add an h-segment σ_{i+1} whose center O_{i+1} is at most 18 edges away from the center O_i of σ_i. ∎

In the next section we show that h-segments that share the same N must yield vertices of $\mathbf{T}$ of high degree.

5. More vertices of high degree

Let σ and σ' be two disjoint h-segments:

$$V_{-5} \overset{e_{-5}}{\text{---}} V_{-4} \cdots V_{-1} \overset{e_{-1}}{\text{---}} O \overset{e_1}{\text{---}} V_1 \cdots V_4 \overset{e_5}{\text{---}} V_5$$

and

$$V'_{-5} \overset{e'_{-5}}{\text{---}} V'_{-4} \cdots V'_{-1} \overset{e'_{-1}}{\text{---}} O' \overset{e'_1}{\text{---}} V'_1 \cdots V'_4 \overset{e'_5}{\text{---}} V'_5$$

together with the corresponding vertices N, N' and labels v, v'. In the remainder of this section we suppose $N' = N$, so that we have $N = O + v = O' + v'$.

Let X' denote the vertex of σ', nearest to O', such that $X' + v' \in \mathbf{S}$. Without loss of generality, suppose $X' \in \{V_1', \cdots, V_5'\}$.

Lemma 5.1. *We have $v' \neq e_1$ and $v' \neq e_{-1}$. Similarly $v \neq e_1'$ and $v \neq e_{-1}'$.*

Proof. We have $V_1 + O' = e_1 + v + v'$. Because V_1 and O' are nonadjacent vertices of $\mathbf{S}$ we must have $v' \neq e_1$. The other cases follow from considering nonadjacent pairs of vertices V_{-1} and O', V_1' and O, V_{-1}' and O. ∎

Lemma 5.2. *Suppose that $d_{\mathbf{S}}(O, O') \geq 13$, so that any vertex of σ' is separated from any vertex of σ by at least three edges of $\mathbf{S}$.*

Let k be the smallest $i \in \{1, \cdots 5\}$, such that $V_i' + v' \in \mathbf{S}$. Then $\{e_1, \cdots, e_k\} \neq \{e_1', \cdots, e_k'\}$, and $\{e_{-1}, \cdots, e_{-k}\} \neq \{e_1', \cdots, e_k'\}$.

Proof. We have $V_k' + v' \in \mathbf{S}$. Consider the path

$$V_k \cdots O — N — O' \cdots V_k' + v'$$

to obtain $V_k + (V_k' + v') = e_1 + \cdots + e_k + v + v' + e_1' + \cdots + e_k' + v'$. Therefore, if $\{e_1, \cdots, e_k\} = \{e_1', \cdots, e_k'\}$, then $V_k + (V_k' + v') = v$, so that V_k and $V_k' + v'$ are adjacent, which contradicts $d_{\mathbf{S}}(O, O') \geq 13$. The other case is analogous. ∎

The following lemma is the main result of this section.

Lemma 5.3. *Suppose that $d_{\mathbf{S}}(O, O') \geq 13$. Suppose $v' + V \in \mathbf{T}$ for all vertices V of σ, and $v + V' \in \mathbf{T}$ for all vertices V' of $[O', X']$. Then there exist four distinct vertices P_1, P_2, P_3, P_4 of $\mathbf{T}$ such that*

- *$P_i + v' \in \sigma$ for $i = 1, 2, 3$ and $P_4 + v \in [O', X']$.*
- *$\sum_{i=1}^{4} \delta(P_i) \geq n$.*

Proof. Consider the translate of σ by v':

$$M_{-5} \overset{e_{-5}}{—} M_{-4} \cdots M_{-1} \overset{e_{-1}}{—} M_0 \overset{e_1}{—} M_1 \cdots M_4 \overset{e_5}{—} M_5$$

where $M_0 = O + v'$, $M_i = V_i + v'$ for $i \in \{-5 \cdots, -1, 1, \cdots, 5\}$. All the vertices M_i, $i \in \{-5, \cdots, +5\}$, are in $\mathbf{T}$.

Notice that $M_0 = O' + v$. Consider $M_1' = V_1' + v \in \mathbf{T}$. We have $M_1' + M_0 = v + e_1' + v = e_1'$, so that M_1' is adjacent to M_0. Figure 2 illustrates the situation.

1. If $e_1' \neq e_1$ and $e_1' \neq e_{-1}$. Then $M_1' \neq M_1$ and $M_1' \neq M_{-1}$, so that the vertices M_{-1}, M_0, M_1, M_1' are distinct. Because M_0 is adjacent to all three others, we have, by Lemma 3.1:

$$\delta(M_{-1}) + \delta(M_0) + \delta(M_1) + \delta(M_1') \geq n.$$

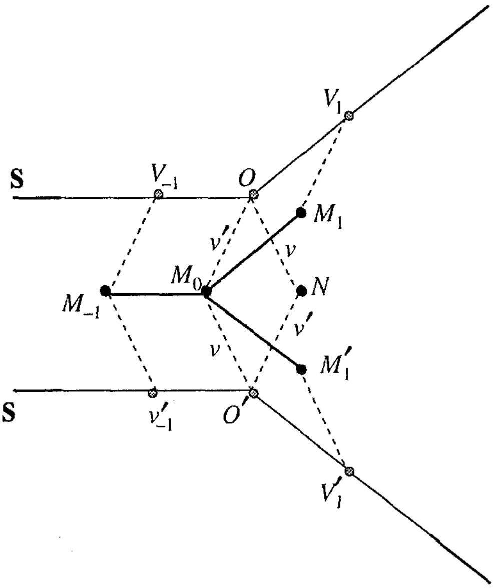

Fig. 2. More vertices of **T** of degree 3

Therefore the result holds with $P_1 = M_{-1}, P_2 = M_0, P_3 = M_1, P_4 = M'_1$.

2. If $e'_1 = e_1$. Consider $M'_2 = V'_2 + v$. We have then $M'_2 + M_1 = v + e'_2 + e'_1 + v + e_1 = e'_2$, so that M'_2 is adjacent to M_1. If $e'_2 \neq e_2$, i.e. $M'_2 \neq M_2$, then M_0, M_1, M_2, M'_2 are distinct vertices and they yield the desired set $\{P_1, P_2, P_3, P_4\}$. If $e'_2 = e_2$, then consider $M'_3 = V'_3 + v$ together with M_1, M_2, M_3 and so on. By Lemma 5.2, this process must stop before we reach $M'_i = V'_i + v$ for $V'_i = X'$.
3. If $e'_1 = e_{-1}$. This time move downwards along the chain of $M_i s$. We have $M'_2 + M_{-1} = v + e'_2 + e'_1 + v + e_{-1} = e'_2$, so that M'_2 is adjacent to M_{-1}. If $e'_2 \neq e_{-2}$, then $M_0, M_{-1}, M_{-2}, M'_2$ are distinct vertices and yield the desired P_1, P_2, P_3, P_4. If $e'_2 = e_{-2}$, look at M'_3 together with M_{-1}, M_{-2}, M_{-3} and so on. Again this process must stop before we consider $M'_i = X' + v$ by Lemma 5.2. ∎

6. Two counting arguments

Let $\mathcal{S}$ be a set of disjoint h-segments of **S**. Let $\mathcal{N}$ be the set of vertices of high degree produced by $\mathcal{S}$, i.e.

$$\mathcal{N} = \bigcup_{\sigma \in \mathcal{S}} \{N(\sigma)\}.$$

It is clear that:

Lemma 6.1. *Let* $\Delta = \sum_{V \in \mathbf{T}} \delta(V)$. *We have:*

$$\Delta \geq |\mathscr{N}| \frac{n}{4}.$$

$\mathscr{N}$ induces the partition of $\mathscr{S}$

$$\mathscr{S} = \bigcup_{N \in \mathscr{N}} \Sigma_N$$

where $\Sigma_N = \{\sigma \in \mathscr{S} \mid N(\sigma) = N\}$.

Now for a given vertex $N \in \mathscr{N}$, let Π_N denote the set of couples (σ, σ') of $\Sigma_N \times \Sigma_N$ such that

- $V + v' \in \mathbf{T}$ for all $V \in \sigma$, and $V' + v \in \mathbf{T}$ for all $V' \in [O', X']$
- $d_{\mathbf{S}}(O, O') \geq 13$.

Lemma 6.2. *We have* $|\Pi_N| \geq |\Sigma_N|(|\Sigma_N| - 16)$.

Proof. Fix $\sigma \in \Sigma_N$. The number of h-segments $\sigma' \in \Sigma_N$ such that $v' \neq v$ and $V + v' \in \mathbf{S}$ for some $V \in \sigma$ is at most the number of different choices of v' that label an edge of $\mathbf{S}$ incident to a vertex of σ. Since $v' \neq e_1, v' \neq e_{-1}$ (Lemma 5.1), and v labels an edge of $\mathbf{S}$ incident to a vertex of σ, this number is at most 9. Hence, the overall number of couples (σ, σ') such that $v' \neq v$ and $V + v' \in \mathbf{S}$ for some $V \in \sigma$ is at most $9|\Sigma_N|$. Similarly, by fixing σ' we obtain that the number of couples (σ, σ') such that $v' \neq v$ and $V' + v \in \mathbf{S}$ for some $V' \in [O', X']$ is at most $4|\Sigma_N|$. Again fixing σ, there are at most two choices of v' that can yield σ' such that $d_{\mathbf{S}}(O, O') < 13$. Hence, $|\Pi_N| \geq |\Sigma_N|(|\Sigma_N| - 1) - 9|\Sigma_N| - 4|\Sigma_N| - 2|\Sigma_N| \geq |\Sigma_N|(|\Sigma_N| - 16)$. ∎

Lemma 5.3 enables us to define, for every $N \in \mathscr{N}$, 4 functions P_1, P_2, P_3, P_4

$$\begin{array}{rccc} P_i : & \Pi_N & \longrightarrow & \mathbf{T} \\ & \pi = (\sigma, \sigma') & \mapsto & P_i(\pi) \end{array}$$

such that, (we shall abuse notation slightly by writing P_i instead of $P_i(\pi)$)

1. For any given $\pi = (\sigma, \sigma') \in \Pi_N$, the vertices $P_i(\pi)$ are distinct for $i = 1, 2, 3, 4$ and
$$\sum_{i=1}^{4} \delta(P_i) \geq n.$$
2. $P_i + v' \in \sigma$ for $i = 1, 2, 3$ and $P_4 + v \in \sigma'$.

The following lemma is the crucial argument of this section. It consists of an evaluation of the total degree Δ of $\mathbf{T}$ through the functions P_i.

Lemma 6.3. *Let* $\Delta = \sum_{V \in \mathbf{T}} \delta(V)$. *We have:*

$$\Delta \geq \sum_{N \in \mathscr{N}} \sum_{\pi \in \Pi_N} \sum_{i=1}^{4} \frac{\delta(P_i(\pi))}{2(n - \delta(P_i(\pi)))}.$$

Proof. The counting procedure consists of adding the contribution of each $P_i(\pi)$, produced by every $\pi = (\sigma, \sigma')$ for every Π_N, to the global degree Δ. The crucial point is that every time a vertex P of $\mathbf{T}$ is produced by some function P_i and a couple (σ, σ'), it comes together with an edge labeled x, with either $x = v$ or $x = v'$, linking it to a vertex of either σ' or σ respectively. Therefore, P together with x determine $\{\sigma, \sigma'\}$ up to ordering. (P and x determine uniquely an h-segment τ such that $P + x \in \tau$, and τ' is determined in turn by $N(\tau') = N(\tau) = N$ and $O(\tau') = N + x$. We must have $\{\sigma, \sigma'\} = \{\tau, \tau'\}$). Hence, since the segments of $\mathscr{S}$ are disjoint, the number of times a given P can be produced by this counting scheme is at most twice the number of edges relating P to $\mathbf{S}$, i.e. $2(n - \delta(P))$. ∎

The rest is now straightforward counting.

Lemma 6.4. *Let* $a = |\mathscr{S}|/|\mathscr{N}|$. *We have:*

$$\Delta \geq \frac{2}{3}|\mathscr{S}|(a - 16).$$

Proof. Let us apply Lemma 6.3. Since $\sum_{i=1}^{4} \delta(P_i) \geq n$, the quantity

$$\sum_{i=1}^{4} \frac{\delta(P_i)}{n - \delta(P_i)}$$

is minimized when $\delta(P_i) = n/4$, i.e.

$$\sum_{i=1}^{4} \frac{\delta(P_i)}{n - \delta(P_i)} \geq \frac{4}{3}$$

whence

$$\Delta \geq \sum_{N \in \mathscr{N}} \frac{2}{3}|\Pi_N| \geq \sum_{N \in \mathscr{N}} \frac{2}{3}|\Sigma_N|(|\Sigma_N| - 16)$$

by Lemma 6.2. Since $\sum_{N \in \mathscr{N}} |\Sigma_N| = |\mathscr{S}|$, the righthandside of the last inequality is minimized for $|\Sigma_N| = |\mathscr{S}|/|\mathscr{N}| = a$ for every $N \in \mathscr{N}$, hence

$$\Delta \geq \frac{2}{3}|\mathscr{N}|a(a - 16)$$

hence the result. ∎

Lemma 6.5. *We have:*

$$\Delta \geq |\mathscr{S}| \left(\frac{n^{1/2}}{\sqrt{6}} - 6 \right).$$

Proof. Lemmas 6.1 and 6.4 together yield:

$$\Delta \geq \min_a \max \left(|\mathscr{S}| \frac{n}{4a} , |\mathscr{S}| \frac{2}{3}(a-16) \right).$$

The minimum in the righthandside of this last inequality is achieved for $a(a-16) = 3n/8$, i.e. for

$$\frac{2}{3}(a-16) = \sqrt{\frac{16^2}{9} + \frac{n}{6}} - \frac{16}{3} \geq \frac{n^{1/2}}{\sqrt{6}} - 6.$$ ∎

Proof of theorem 1.1. Apply Lemmas 4.1 and 6.5. We obtain

$$\Delta \geq \frac{|\mathbf{S}| - 10}{18} \left(\frac{n^{1/2}}{\sqrt{6}} - 6 \right),$$

which together with Lemma 2.1 yields

$$n2^{n-1} \geq |\mathbf{S}| \left[(n-1) + \frac{1}{36} \left(\frac{n^{1/2}}{\sqrt{6}} - 6 \right) \right] - \frac{5}{18} \left(\frac{n^{1/2}}{\sqrt{6}} - 6 \right).$$

The result follows after routine adjustments. ∎

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